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Uniqueness of Solutions for the Ginzburg–Landau Model of Superconductivity in Three Spatial Dimensions

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This paper deals with the time-dependent Ginzburg–Landau equations of superconductivity in three spatial dimensions. The uniqueness of global weak solutions is established for this model with initial data of the order parameter in L^4 and magnetic potential in L^3.

Key Words: superconductivity; uniqueness; Ginzburg–Landau equation; global weak solution.

1. INTRODUCTION

In the present paper, we establish the uniqueness of weak solutions to the time-dependent Ginzburg–Landau (TDGL for short) model of superconductivity in three spatial dimensions. The TDGL equations characterize the behaviour of superconductivity materials and have been used to study, both analytically and numerically, the motion and interaction of vortex-like structures in superconductors. This model involves three unknown functions. A complex valued function $\psi\colon \Omega \to C$ is the order parameter; a vector valued function $A\colon \Omega \to R^d$ is the magnetic potential; a scalar valued function $\phi\colon \Omega \to R$ is the electric potential. Here the domain Ω is a smooth bounded domain in R^d, $d = 2, 3$. The nondimensional TDGL equations take the form

$$\eta\left(\frac{\partial}{\partial t} + i\kappa\phi\right)\psi + \left(\frac{i}{\kappa}\operatorname{grad} + A\right)^2\psi + \left(|\psi|^2 - 1\right)\psi = 0, \qquad (1.1)$$

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$$\frac{\partial A}{\partial t} + \operatorname{grad} \phi + \operatorname{curl}^2 A - \frac{1}{2i\kappa}(\psi^* \operatorname{grad} \psi - \psi \operatorname{grad} \psi^*) + |\psi|^2 A - \operatorname{curl} H = 0, \tag{1.2}$$

where η is the nondimensional diffusivity, and κ is the Ginzburg–Landau parameter. Also, H is the applied magnetic field, and ψ^* is the complex conjugate of ψ. For physical background about the TDGL equations we refer the readers to [1–3].

Obviously, the TDGL model given above is not well posed, since there are three unknown functions in the two coupled equations. However, this model possesses a gauge invariant property from which we can find a third equation. This gauge invariance can be stated as follows: if (ψ, ϕ, A) is a solution of system (1.1), (1.2), then for any smooth function θ,

$$\left(\psi e^{i\kappa\theta}, \phi - \theta_t, A + \operatorname{grad} \theta\right)$$

is also a solution. Therefore, by suitably choosing the function θ, we can get a relation between the unknown function ψ and A; see, e.g., [4–9] for more details.

With suitable gauge choices, various properties of solutions for the TDGL model have been investigated in many works such as [4–15] and the references therein. In particular, the existence and uniqueness of solutions for systems (1.1), (1.2) have been studied in [4–6]. In these articles, the authors assumed that the initial data ψ_0 of the order parameter satisfied $\|\psi_0\|_{L^\infty} \leq 1$. In [14], Tang and Wang removed this condition and proved the existence and uniqueness of global solutions in H^1 space. Recently, Wang and Zhan [15] established the well-posedness of solutions in L^2 when space dimension $d = 2$ and the existence of weak solutions in L^p with $p \geq 4$ for space dimension $d = 3$. In [16], Wang and Su improved this result to L^p with $p \geq 3$ when $d = 3$. However, the uniqueness of the L^p solutions for the TDGL equations remains open in [15, 16] for $d = 3$. In this paper, we first establish the L^p regularity of solutions and then, based on the regularity, show the uniqueness of weak solutions for the periodic TDGL equations with any L^p $(p \geq 4)$ initial data. In fact, our result is valid for any initial data ψ_0 of order parameter in L^p with $p \geq 4$, and A_0 of magnetic potential in L^p with $p \geq 3$. The uniqueness results rely on detailed a priori estimates on the solutions which will be derived in Section 3.

The outline of this paper is as follows. In Section 2, we state our main result, that is, the uniqueness of weak solutions in L^p space. Section 3 contains detailed a priori estimates on solutions. We first establish a priori bounds on the order parameter ψ and then on the magnetic potential A. In Section 4, we present the proof of our main result.

2. THE MAIN RESULT

In this section, we consider the TDGL equations with a fixed gauge. Here we adopt the Lorentz gauge $\phi = -\operatorname{div} A$ (see, e.g., [4, 5]); then Eqs. (1.1) and (1.2) reduce to a system of equations for ψ and A:

$$\eta \frac{\partial \psi}{\partial t} + \left(\frac{i}{\kappa} \operatorname{grad} + A \right)^2 \psi = i\eta\kappa\psi \operatorname{div} A + \left(1 - |\psi|^2\right)\psi, \qquad \text{in } \Omega \times (0, \infty), \tag{2.1}$$

$$\frac{\partial A}{\partial t} - \Delta A = \frac{1}{2i\kappa}(\psi^* \operatorname{grad} \psi - \psi \operatorname{grad} \psi^*) - |\psi|^2 A + \operatorname{curl} H, \quad \text{in } \Omega \times (0, \infty). \tag{2.2}$$

The system (2.1), (2.2) is supplemented with the periodic boundary conditions

$$\psi \text{ and } A \text{ are } \Omega\text{-periodic}, \qquad \Omega = \prod_{i=1}^{3} (0, L_i), \tag{2.3}$$

and the initial conditions

$$\psi(x, 0) = \psi_0(x), \quad A(x, 0) = A_0(x), \qquad \text{in } \Omega. \tag{2.4}$$

In the sequel, we denote by $W_{\text{per}}^{s,p}(\Omega)$ the subspaces of the standard Sobolev spaces which consist of periodic real scalar (or vector) valued functions defined on Ω, and as usual, $W_{\text{per}}^{s,2}(\Omega)$ is denoted by $H_{\text{per}}^{s}(\Omega)$. Sobolev spaces of complex valued functions are denoted by $\mathscr{W}_{\text{per}}^{s,p}(\Omega)$ and $\mathscr{H}_{\text{per}}^{s}(\Omega)$ with calligraphic letters. We use $\|\cdot\|$ and $(\cdot,\cdot)$ for the usual norm and inner product of $L_{\text{per}}^2(\Omega)$ (or $\mathscr{L}_{\text{per}}^2(\Omega)$) respectively; $\forall 1 \le p \le \infty$, denote by $\|\cdot\|_p$ the norm of $L_{\text{per}}^p(\Omega)(\|\cdot\|_2 = \|\cdot\|)$. $\|\cdot\|_X$ denotes the norm of Banach space X.

We now introduce the weak form of the TDGL equations as follows.

DEFINITION 2.1. Assume that $(\psi_0, A_0) \in \mathscr{L}_{\text{per}}^2 \times L_{\text{per}}^2$. Then a pair of functions $(\psi(t), A(t))$ $(t \ge 0)$ is called a weak solution of problem (2.1)–(2.4), if for every $T > 0$,

$$\psi(t) \in L^\infty\left(0, T; \mathscr{L}_{\text{per}}^2\right) \cap L^2\left(0, T; \mathscr{H}_{\text{per}}^1\right),$$

$$A(t) \in L^\infty\left(0, T; L_{\text{per}}^2\right) \cap L^2\left(0, T; H_{\text{per}}^1\right),$$

and the following holds in the sense of distribution:

$$\eta \frac{d}{dt} \int_\Omega \psi \tilde{\psi}^* \, dx + \int_\Omega \left(\frac{i}{\kappa} \operatorname{grad} + A \right) \psi \cdot \left(-\frac{i}{\kappa} \operatorname{grad} + A \right) \tilde{\psi}^* \, dx$$

$$= i\eta\kappa \int_\Omega \psi \tilde{\psi}^* \operatorname{div} A \, dx + \int_\Omega \left(1 - |\psi|^2 \right) \psi \tilde{\psi}^* \, dx, \qquad \forall \tilde{\psi} \in \mathscr{H}^1_{\mathrm{per}}, \quad (2.5)$$

$$\frac{d}{dt} \int_\Omega A \tilde{A} \, dx + \int_\Omega \operatorname{grad} A \cdot \operatorname{grad} \tilde{A} \, dx$$

$$= \frac{1}{2i\kappa} \int_\Omega \left(\psi^* \operatorname{grad} \psi - \psi \operatorname{grad} \psi^* \right) \tilde{A} \, dx$$

$$- \int_\Omega |\psi|^2 A \tilde{A} \, dx + \int_\Omega \operatorname{curl} H \cdot \tilde{A} \, dx, \qquad \forall \tilde{A} \in H^1_{\mathrm{per}}, \quad (2.6)$$

$$\left(\psi(0), \tilde{\psi} \right) = \left(\psi_0, \tilde{\psi} \right), \qquad \forall \tilde{\psi} \in \mathscr{L}^2_{\mathrm{per}}, \quad (2.7)$$

$$\left(A(0), \tilde{A} \right) = \left(A_0, \tilde{A} \right), \qquad \forall \tilde{A} \in L^2_{\mathrm{per}}(\Omega). \quad (2.8)$$

Then, we have the following main result.

THEOREM 2.1. *Assume that* $\psi_0 \in \mathscr{L}^4_{\mathrm{per}}(\Omega)$, $A_0 \in L^3_{\mathrm{per}}(\Omega)$, $H \in L^\infty(0, \infty; H^1_{\mathrm{per}}(\Omega))$. *Then there exists a unique global weak solution* $(\psi(t), A(t))$ *for problem* (2.1)–(2.4) *such that*

$$\psi(t) \in L^\infty\left(0, T; \mathscr{L}^4_{\mathrm{per}}(\Omega)\right) \cap L^6\left(0, T; \mathscr{L}^6_{\mathrm{per}}(\Omega)\right) \cap L^2\left(0, T; \mathscr{H}^1_{\mathrm{per}}(\Omega)\right),$$

$$A(t) \in L^\infty\left(0, T; L^3_{\mathrm{per}}(\Omega)\right) \cap L^2\left(0, T; H^1_{\mathrm{per}}(\Omega)\right).$$

Moreover, if $A_0 \in L^4_{\mathrm{per}}(\Omega)$, *then*

$$A(t) \in L^\infty\left(0, T; L^4_{\mathrm{per}}(\Omega)\right) \cap L^2\left(0, T; H^1_{\mathrm{per}}(\Omega)\right).$$

We remark that the existence of global weak solutions for the TDGL equations with L^p initial data was shown for $p \geq 4$ in [15] and for $p \geq 3$ in [16], respectively. Therefore, in the sequel, we only need to show the uniqueness of L^p weak solutions.

For our purpose, we recall the following interpolation inequality (see [17]):

LEMMA 2.1. *Let* $\Omega \subset R^n$ *be a bounded smooth domain,* $u \in W^{m,r}(\Omega) \cap L^q(\Omega)$, $0 \leq l \leq m$, *and*

$$\frac{l}{m} \leq \alpha < 1, \quad 1 < r, \quad q < \infty, \qquad \frac{1}{p} = \frac{l}{n} + \alpha \left(\frac{1}{r} - \frac{m}{n} \right) + \frac{1 - \alpha}{q}.$$

Then the inequality

$$\|D^l u\|_p \le C\|u\|_{W^{m,r}}^{\alpha}\|u\|_q^{1-\alpha},$$

where $C = C(\Omega, n, m, l, q, r, \alpha)$ *is a constant, holds.*

3. A PRIORI ESTIMATES

In this section, we derive a priori estimates on solutions of problems (2.1)–(2.4) which are needed to show the uniqueness of solutions in the next section. We first establish a priori bounds for the order parameter ψ, and then for the magnetic potential A. We begin with the following lemma.

LEMMA 3.1. *Assume that* $\psi_0 \in \mathscr{L}^4_{\text{per}}$; *then* $\forall T > 0$, *we have*

$$\int_0^T\int_\Omega |\psi|^6\,dx\,dt + \int_0^T\int_\Omega |\psi|^2\left|\left(\frac{i}{\kappa}\operatorname{grad} + A\right)\psi\right|^2 dx\,dt \le C(T),$$

$$\|\psi(t)\|_4 \le C(T), \qquad \forall 0 \le t \le T,$$

where $C(T)$ *depends on* T *and* $\|\psi_0\|_4$.

Proof. Taking the real part of the $\mathscr{L}^2$ inner product of (2.1) with $|\psi|^2\psi$, we find that

$$\frac{1}{4}\eta\frac{d}{dt}\int_\Omega |\psi|^4\,dx + \operatorname{Re}\int_\Omega\left(\frac{i}{\kappa}\operatorname{grad} + A\right)\psi\cdot\left(-\frac{i}{\kappa}\operatorname{grad} + A\right)|\psi|^2\psi^*\,dx$$
$$+\int_\Omega |\psi|^6\,dx = \int_\Omega |\psi|^4\,dx. \tag{3.1}$$

Note that

$$\operatorname{Re}\int_\Omega\left(\frac{i}{\kappa}\operatorname{grad} + A\right)\psi\cdot\left(-\frac{i}{\kappa}\operatorname{grad} + A\right)|\psi|^2\psi^*\,dx$$
$$= \int_\Omega |\psi|^2\left|\left(-\frac{i}{\kappa}\operatorname{grad} + A\right)\psi\right|^2 dx$$
$$+ \operatorname{Re}\int_\Omega\left(\frac{i}{\kappa}\operatorname{grad} + A\right)\psi\cdot\left(-\frac{i}{\kappa}\psi^*\operatorname{grad}|\psi|^2\right)dx$$
$$= \int_\Omega |\psi|^2\left|\left(\frac{i}{\kappa}\operatorname{grad} + A\right)\psi\right|^2 dx + \frac{1}{\kappa^2}\operatorname{Re}\int_\Omega \psi^*\operatorname{grad}\psi\cdot\operatorname{grad}|\psi|^2\,dx$$

$$= \int_\Omega |\psi|^2 \left| \left(\frac{i}{\kappa} \operatorname{grad} + A \right) \psi \right|^2 dx + \frac{1}{\kappa^2} \int_\Omega |\psi|^2 |\operatorname{grad} \omega|^2 \, dx$$

$$+ \frac{1}{\kappa^2} \operatorname{Re} \int_\Omega (\psi^*)^2 (\operatorname{grad} \psi)^2 \, dx$$

$$\geq \int_\Omega |\psi|^2 \left| \left(\frac{i}{\kappa} \operatorname{grad} + A \right) \psi \right|^2 dx. \tag{3.2}$$

The last inequality is obtained by

$$|\psi|^2 |\operatorname{grad} \psi|^2 + \operatorname{Re}(\psi^*)^2 (\operatorname{grad} \psi)^2 \geq 0.$$

Due to

$$|\psi|^4 = 1 \cdot |\psi|^4 \leq \frac{2}{3} |\psi|^6 + \frac{1}{3},$$

we get that

$$\int_\Omega |\psi|^4 \, dx \leq \frac{2}{3} \int_\Omega |\psi|^6 \, dx + \frac{1}{3} |\Omega|. \tag{3.3}$$

It follows from (3.1)–(3.3) that

$$\frac{1}{4} \eta \frac{d}{dt} \int_\Omega |\psi|^4 \, dx + \int_\Omega |\psi|^2 \left| \left(\frac{i}{\kappa} \operatorname{grad} + A \right) \psi \right|^2 dx + \frac{1}{3} \int_\Omega |\psi|^6 \, dx \leq \frac{1}{3} |\Omega|. \tag{3.4}$$

Integrating (3.4) between 0 and T, we obtain that

$$\| \psi(t) \|_4^4 \leq \frac{4}{3} \frac{|\Omega| T}{\eta} + \| \psi_0 \|_4^4, \qquad \forall 0 \leq t \leq T,$$

$$\int_0^T \int_\Omega |\psi|^2 \left| \left(\frac{i}{\kappa} \operatorname{grad} + A \right) \psi \right|^2 dx \leq \frac{1}{3} |\Omega| T + \frac{\eta}{4} \| \psi_0 \|_4^4,$$

$$\int_0^T \int_\Omega |\psi|^6 \, dx \leq |\Omega| T + \frac{3\eta}{4} \| \psi_0 \|_4^4,$$

which concludes the proof of Lemma 3.1.

LEMMA 3.2. *Assume that* $\psi_0 \in \mathscr{L}^4_{\text{per}}(\Omega)$, $A_0 \in L^2_{\text{per}}(\Omega)$; *then* $\forall T > 0$, *there exists a constant* $C(T)$ *depending on* T *and* $\|\psi_0\|_4$ *and* $\|A_0\|$ *such that*

$$\|A(t)\| \le C(T), \qquad \forall 0 \le t \le T,$$

$$\int_0^T \|\text{grad } A\|^2 \, dt \le C(T).$$

Proof. Taking the inner product of (2.2) with A in L^2, we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|A\|^2 + \|\text{grad } A\|^2 \\
&\quad = -\text{Re} \int_\Omega \psi^* \left(\frac{i}{\kappa} \text{ grad } + A \right) \psi \cdot A \, dx + \int_\Omega \text{curl } H \cdot A \, dx \\
&\quad \le \int_\Omega |\psi| \left| \left(\frac{i}{\kappa} \text{ grad } + A \right) \psi \right| |A| \, dx + \|\text{curl } H\| \, \|A\| \\
&\quad \le \frac{1}{2}\|A\|^2 + \int_\Omega |\psi|^2 \left| \left(\frac{i}{\kappa} \text{ grad } + A \right) \psi \right|^2 dx + \|\text{curl } H\|^2,
\end{aligned}$$

so we get that

$$\begin{aligned}
&\frac{d}{dt}\|A\|^2 + 2\|\text{grad } A\|^2 \\
&\quad \le \|A\|^2 + 2\int_\Omega |\psi|^2 \left| \left(\frac{i}{\kappa} \text{ grad } A \right) \psi \right|^2 dx + 2\|\text{curl } H\|^2. \qquad (3.5)
\end{aligned}$$

By the Gronwall lemma we infer that

$$\begin{aligned}
\| A(t) \|^2 &\le \| A(0) \|^2 e^t \\
&\qquad + 2\int_0^t \int_\Omega | \psi(s) |^2 \left| \left(\frac{i}{\kappa} \text{ grad } + A(s) \right) \psi(s) \right|^2 e^{t-s} \, ds \\
&\qquad + 2\int_0^t \| \text{curl } H(s) \|^2 e^{t-s} \, ds \\
&\le C(T), \qquad \forall 0 \le t \le T. \qquad (3.6)
\end{aligned}$$

The last inequality is obtained by Lemma 3.1. Integrating (3.5) and using (3.6) we have

$$\int_0^T \|\text{grad } A\|^2 \, dt \le C(T). \tag{3.7}$$

(3.6) and (3.7) complete the proof.

LEMMA 3.3. *Assume that* $\psi_0 \in \mathscr{L}^4_{\text{per}}(\Omega)$, $A_0 \in L^3_{\text{per}}(\Omega)$; *then* $\forall T > 0$, *there exists a constant* $C(T)$ *depending on* T *and* $\|\psi_0\|_4$ *and* $\|A_0\|_3$ *such that*

$$\|A(t)\|_3 \le C(T), \qquad \forall 0 \le t \le T,$$

$$\sum_{i=1}^{3} \int_0^T \int_\Omega |\text{grad}|A_i|^{3/2}|^2 \, dx \, dt \le C(T),$$

where $A = (A_1, A_2, A_3)$.

Proof. We shall denote by C any positive constants hereafter. Let $n \ge 1$ and $\tilde{A} = (|A_1|^{2n/(2n-1)} A_1, |A_2|^{2n/(2n-1)} A_2, |A_3|^{2n/(2n-1)} A_3)$. Then taking the inner product of (2.2) with $\tilde{A}$ in L^2, we see that

$$\frac{2n-1}{2(3n-1)} \frac{d}{dt} \int_\Omega \sum_{i=1}^{3} |A_i|^{2+2n/(2n-1)} \, dx - \sum_{i=1}^{3} \int_\Omega \Delta A_i \cdot |A_i|^{2n/(2n-1)} A_i \, dx$$
$$= -\int_\Omega \text{Re}\left(\psi^* \left(\frac{i}{\kappa} \text{grad} + A \right) \psi \right) \tilde{A} \, dx + \int_\Omega \text{curl } H \cdot \tilde{A} \, dx. \tag{3.8}$$

Note that

$$-\sum_{i=1}^{3} \int_\Omega \Delta A_i \cdot |A_i|^{2n/(2n-1)} A_i \, dx$$
$$= \sum_{i=1}^{3} \frac{4n-1}{2n-1} \int_\Omega |A_i|^{2n/(2n-1)} |\text{grad } A_i|^2 \, dx, \tag{3.9}$$

and

$$\left| -\int_\Omega \text{Re}\left(\psi^* \left(\frac{i}{\kappa} \text{grad} + A \right) \psi \right) \tilde{A} \, dx \right|$$
$$\le \int_\Omega \left| \psi^* \left(\frac{i}{\kappa} \text{grad} + A \right) \psi \right| |\tilde{A}| \, dx$$
$$\le \int_\Omega |\psi| \left| \left(\frac{i}{\kappa} \text{grad} + A \right) \psi \right| |A|^{1+2n/(2n-1)} \, dx$$
$$\le \frac{1}{2} \int_\Omega |\psi|^2 \left| \left(\frac{i}{\kappa} \text{grad} + A \right) \psi \right|^2 dx + \frac{1}{2} \int_\Omega |A|^{2+4n/(2n-1)} \, dx. \tag{3.10}$$

By the interpolation inequality, we have

$$\int_\Omega |A|^{2+4n/(2n-1)}\,dx$$
$$\le C\|A\|_{H^1}^{6n/(3n-2)}\|A\|_{(6n-2)/(2n-1)}^{4(n-1)(3n-1)/(2n-1)(3n-2)}$$
$$\le \frac{nC}{3n-2}\|A\|_{H^1}^{6n/(3n-2)} + \frac{2(n-1)C}{3n-2}\|A\|_{H^1}^{6n/(3n-2)}\|A\|_{(6n-2)/(2n-1)}^{(6n-2)/(2n-1)}. \tag{3.11}$$

The last inequality follows from

$$\|A\|_{(6n-2)/(2n-1)}^{4(n-1)(3n-1)/(2n-1)(3n-2)} \le \frac{n}{3n-2} + \frac{2(n-1)}{3n-2}\|A\|_{(6n-2)/(2n-1)}^{(6n-2)/(2n-1)}.$$

By (3.10) and (3.11) we get that

$$\left|-\int_\Omega \operatorname{Re}\left(\psi^*\left(\frac{i}{\kappa}\operatorname{grad} + A\right)\psi\right)\tilde{A}\,dx\right|$$
$$\le \frac{1}{2}\int_\Omega |\psi|^2\left|\left(\frac{i}{\kappa}\operatorname{grad} + A\right)\psi\right|^2 dx + \frac{nC}{3n-2}\|A\|_{H^1}^{6n/(3n-2)}$$
$$+ \frac{2(n-1)C}{3n-2}\|A\|_{H^1}^{6n/(3n-2)}\|A\|_{(6n-2)/(2n-1)}^{(6n-2)/(2n-1)}. \tag{3.12}$$

Similarly, we have

$$\int_\Omega |\operatorname{curl} H\cdot\tilde{A}|\,dx \le \int_\Omega |\operatorname{curl} H|\,|A|^{1+2n/(2n-1)}\,dx$$
$$\le \frac{1}{4}\|\operatorname{curl} H\|^2 + \int_\Omega |A|^{2+4n/(2n-1)}\,dx$$
$$\le \frac{1}{4}\|\operatorname{curl} H\|^2 + \frac{nC}{3n-2}\|A\|_{H^1}^{6n/(3n-2)}$$
$$+ \frac{2(n-1)C}{3n-2}\|A\|_{H^1}^{6n/(3n-2)}\|A\|_{(6n-2)/(2n-1)}^{(6n-2)/(2n-1)}. \tag{3.13}$$

By (3.8), (3.9), (3.12), and (3.13) we find that

$$\frac{2n-1}{2(3n-1)}\frac{d}{dt}\|A\|_{(6n-2)/(2n-1)}^{(6n-2)/(2n-1)}+\sum_{i=1}^{3}\frac{4n-1}{2n-1}\int_\Omega |A_i|^{2n/(2n-1)}|\text{grad } A_i|^2\,dx$$
$$\leq C\int_\Omega |\psi|^2\left|\left(\frac{i}{\kappa}\text{grad}+A\right)\psi\right|^2 dx+\frac{nC}{3n-2}\|A\|_{H^1}^{6n/(3n-2)}$$
$$+\frac{1}{4}\|\text{curl } H\|^2+\frac{2(n-1)C}{3n-2}\|A\|_{H^1}^{6n/(3n-2)}\|A\|_{(6n-2)/(2n-1)}^{(6n-2)/(2n-1)}. \quad (3.14)$$

Multiplying (3.14) by $\frac{2(3n-1)}{2n-1}$, and noting that every coefficient of the resulting inequality is bounded with respect to n, we claim that there exists a constant C independent of n such that

$$\frac{d}{dt}\|A\|_{(6n-2)/(2n-1)}^{(6n-2)/(2n-1)}\leq C\int_\Omega |\psi|^2\left|\left(\frac{i}{\kappa}\text{grad}+A\right)\psi\right|^2 dx$$
$$+C\|A\|_{H^1}^{6n/(3n-2)}+C\|\text{curl } H\|^2$$
$$+C\|A\|_{H^1}^{6n/(3n-2)}\|A\|_{(6n-2)/(2n-1)}^{(6n-2)/(2n-1)}.$$

By the Gronwall lemma, we get that

$$\|A(t)\|_{(6n-2)/(2n-1)}^{(6n-2)/(2n-1)}\leq \|A_0\|_{(6n-2)/(2n-1)}^{(6n-2)/(2n-1)}\exp\left(\int_0^t C\|A(t)\|_{H^1}^{6n/(3n-2)}\,dt\right)$$
$$+\int_0^t g_n(s)\exp\left(\int_s^t C\|A(\tau)\|_{H^1}^{6n/(3n-2)}\,d\tau\right)ds, \quad (3.15)$$

where

$$g_n(t)=C\int_\Omega |\psi|^2\left|\left(\frac{i}{\kappa}\text{grad}+A\right)\psi\right|^2 dx+C\|A\|_{H^1}^{6n/(3n-2)}+C\|\text{curl } H\|^2.$$

Taking the limit of (3.15) as $n\to\infty$, we see that

$$\|A(t)\|_3^3\leq \|A_0\|_3^3\exp\left(\int_0^t \|A(t)\|_{H^1}^2\,dt\right)$$
$$+\int_0^t g(s)\exp\left(\int_s^t C\|A(\tau)\|_{H^1}^2\,d\tau\right)ds$$
$$\leq C\left(1+\|A_0\|_3^3\right),\qquad \forall 0\leq t\leq T, \quad (3.16)$$

where

$$g(t) = C\int_\Omega |\psi|^2 \left| \left(\frac{i}{\kappa} \operatorname{grad} + A \right) \psi \right|^2 dx + C\|A\|_{H^1}^2 + C\|\operatorname{curl} H\|^2,$$

and the last inequality follows from Lemmas 3.1 and 3.2. Again, integrating (3.14) between 0 and T, then letting $n \to \infty$ and using (3.16), we find that

$$\sum_{i=1}^{3} \int_0^T \int_\Omega |A_i| \, |\operatorname{grad} A_i|^2 \, dx \, dt \le C.$$

Due to

$$|\operatorname{grad} |A_i|^{3/2}| = \frac{3}{2} |A_i|^{1/2} |\operatorname{grad} A_i|,$$

it follows that

$$\sum_{i=1}^{3} \int_0^T \int_\Omega |\operatorname{grad} |A_i|^{3/2}|^2 \, dx \, dt \le C. \tag{3.17}$$

(3.16) and (3.17) conclude the proof of Lemma 3.3.

LEMMA 3.4. *Assume that* $\psi_0 \in \mathscr{L}^4_{\text{per}}(\Omega)$, $A_0 \in L^4_{\text{per}}(\Omega)$; *then* $\forall T > 0$, *there exists a constant* $C(T)$ *depending on* T, $\|\psi_0\|_4$, *and* $\|A_0\|_4$ *such that*

$$\|A(t)\|_4 \le C(T), \qquad \forall 0 \le t \le T,$$

$$\sum_{i=1}^{3} \int_0^T \int_\Omega |\operatorname{grad} |A_i|^2|^2 \, dx \, dt \le C(T).$$

Proof. Let $\tilde{A} = (|A_1|^2 A_1, \ |A_2|^2 A_2, \ |A_3|^2 A_3)$. Then taking the inner product of (2.2) with $\tilde{A}$ in L^2, we find that

$$\begin{aligned} \frac{1}{4} \frac{d}{dt} \int_\Omega \sum_{i=1}^{3} |A_i|^4 \, dx - \sum_{i=1}^{3} \int_\Omega \Delta A_i \cdot |A_i|^2 A_i \, dx \\ = -\int_\Omega \operatorname{Re}\left(\psi^* \left(\frac{i}{\kappa} \operatorname{grad} + A \right) \psi \right) \tilde{A} \, dx + \int_\Omega \operatorname{curl} H \cdot \tilde{A} \, dx. \end{aligned} \tag{3.18}$$

Obviously,

$$-\sum_{i=1}^{3}\int_\Omega \Delta A_i \cdot |A_i|^2 A_i\, dx = 3\sum_{i=1}^{3}\int_\Omega |A_i|^2\, |\text{grad } A_i|^2\, dx$$
$$= \frac{3}{4}\sum_{i=1}^{3}\int_\Omega |\text{grad}|A_i|^2|^2\, dx, \tag{3.19}$$

and

$$\left|-\int_\Omega \text{Re}\left(\psi^*\left(\frac{i}{\kappa}\text{ grad} + A\right)\psi\right)\tilde{A}\, dx\right|$$
$$\le \frac{1}{4}\int_\Omega |\psi|^2\left|\left(\frac{i}{\kappa}\text{ grad} + A\right)\psi\right|^2 dx + \sum_{i=1}^{3}\int_\Omega |A_i|^6\, dx. \tag{3.20}$$

To estimate the above inequality, for fixed $i = 1, 2, 3$, we set $v = |A_i|^{3/2}$, $u = |A_i|^2$. Then we see that

$$\int_\Omega |A_i|^4\, dx = \int_\Omega v^{8/3}\, dx, \qquad \int_\Omega |A_i|^6\, dx = \int_\Omega u^3\, dx. \tag{3.21}$$

Since

$$\|v\|_{8/3} \le C\|v\|_{H^1}^{3/8}\|v\|^{5/8},$$

we infer that

$$\int_\Omega |A_i|^4\, dx \le C\|v\|_{H^1}\|v\|^{5/3} \le C(\|v\| + \|\text{grad } v\|)\|v\|^{5/3}$$
$$\le C\left(\|A_i\|_3^{3/2} + \|\text{grad}|A_i|^{3/2}\|\right)\|A_i\|_3^{5/2} \le C\|\text{grad}|A_i|^{3/2}\| + C$$
$$\text{(by Lemma 3.3)}. \tag{3.22}$$

Due to

$$\|u\|_3 \le C\|u\|_{H^1}^{1/2}\|u\|^{1/2} \le C\|u\| + C\|\text{grad } u\|^{1/2}\|u\|^{1/2},$$

we see that

$$\int_\Omega |A_i|^6\, dx \le C\|u\|^3 + C\|\text{grad } u\|^{3/2}\|u\|^{3/2} \le \frac{1}{4}\|\text{grad } u\|^2 + C\|u\|^6 + C$$
$$\le \frac{1}{4}\int_\Omega \|\text{grad}|A_i|^2|^2\, dx + C\left(\int_\Omega |A_i|^4\, dx\right)^3 + C$$
$$\le \frac{1}{4}\int_\Omega \|\text{grad}|A_i|^2|^2\, dx$$
$$+ C\left(1 + \|\text{grad}|A_i|^{3/2}\|^2\right)\int_\Omega |A_i|^4\, dx + C. \tag{3.23}$$

The last inequality is obtained by (3.22). From (3.20) and (3.23) we have

$$\begin{aligned}
&\left| -\int_\Omega \operatorname{Re}\left(\psi^* \left(\frac{i}{\kappa} \operatorname{grad} + A \right) \psi \right) \tilde{A}\, dx \right| \\
&\quad \le \frac{1}{4} \int_\Omega |\psi|^2 \left| \left(\frac{i}{\kappa} \operatorname{grad} + A \right) \psi \right|^2 dx + \frac{1}{4} \sum_{i=1}^{3} \int_\Omega |\operatorname{grad} |A_i|^2|^2\, dx \\
&\qquad + C \sum_{i=1}^{3} \left(1 + \|\operatorname{grad} |A_i|^{3/2}\|^2 \right) \int_\Omega |A_i|^4\, dx + C \\
&\quad \le \frac{1}{4} \int_\Omega |\psi|^2 \left| \left(\frac{i}{\kappa} \operatorname{grad} + A \right) \psi \right|^2 dx + \frac{1}{4} \sum_{i=1}^{3} \int_\Omega |\operatorname{grad} |A_i|^2|^2\, dx \\
&\qquad + C \sum_{i=1}^{3} \left(1 + \|\operatorname{grad} |A_i|^{3/2}\|^2 \right) \int_\Omega \sum_{i=1}^{3} |A_i|^4\, dx + C. \qquad (3.24)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\left| \int_\Omega \operatorname{curl} H \cdot \tilde{A}\, dx \right| &\le \frac{1}{4} \|\operatorname{curl} H\|^2 + \int_\Omega \sum_{i=1}^{3} |A_i|^6\, dx \\
&\le \frac{1}{4} \|\operatorname{curl} H\|^2 + \frac{1}{4} \sum_{i=1}^{3} \int_\Omega |\operatorname{grad} |A_i|^2|^2\, dx \\
&\qquad + C \sum_{i=1}^{3} \left(1 + \|\operatorname{grad} |A_i|^{3/2}\|^2 \right) \int_\Omega \sum_{i=1}^{3} |A_i|^4\, dx + C. \qquad (3.25)
\end{aligned}$$

From (3.18), (3.19), (3.24), and (3.25) we claim that

$$\begin{aligned}
&\frac{d}{dt} \int_\Omega \sum_{i=1}^{3} |A_i|^4\, dx + \sum_{i=1}^{3} \int_\Omega |\operatorname{grad} |A_i|^2|^2\, dx \\
&\quad \le \int_\Omega |\psi|^2 \left| \left(\frac{i}{\kappa} \operatorname{grad} + A \right) \psi \right|^2 dx + \|\operatorname{curl} H\|^2 \\
&\qquad + C \sum_{i=1}^{3} \left(1 + \|\operatorname{grad} |A_i|^{3/2}\|^2 \right) \int_\Omega \sum_{i=1}^{3} |A_i|^4\, dx + C. \qquad (3.26)
\end{aligned}$$

By the Gronwall lemma we infer that

$$
\begin{aligned}
\| A(t) \|_4^4 \leq \|A_0\|_4^4 \exp\left(C \sum_{i=1}^{3} \int_0^t \left(1 + \|\mathrm{grad}|A_i|^{3/2}\|^2\right) dt \right) \\
+ \int_0^t \left(C + \| \mathrm{curl}\, H(s) \|^2 \right) \\
\times \exp\left(C \sum_{i=1}^{3} \int_s^t \left(1 + \|\mathrm{grad}|A_i|^{3/2}\|^2\right) d\tau \right) ds \\
+ \int_0^t \int_\Omega | \psi(s) |^2 \left| \left(\frac{i}{\kappa} \mathrm{grad} + A(s) \right) \psi(s) \right|^2 \\
\times \exp\left(C \sum_{i=1}^{3} \int_s^t \left(1 + \|\mathrm{grad}|A_i|^{3/2}\|^2\right) d\tau \right) ds\, dx \\
\leq C\left(1 + \|A_0\|_4^4\right), \qquad \forall 0 \leq t \leq T.
\end{aligned}
\tag{3.27}
$$

The last inequality follows from Lemmas 3.1 and 3.3. Integrating (3.26) between 0 and T, then using (3.27) and Lemmas 3.1 and 3.3 again, we get that

$$
\sum_{i=1}^{3} \int_0^T \int_\Omega |\mathrm{grad}|A_i|^2|^2 \Big)\, dx\, dt \leq C. \tag{3.28}
$$

(3.27) and (3.28) imply Lemma 3.4.

4. PROOF OF THE MAIN RESULT

In this section, we will use the estimates in Section 3 and present the proof of Theorem 2.1. In the sequel, we assume that (ψ_1, A_1) and (ψ_2, A_2) are two solutions of problem (2.1)–(2.4) and set $\psi = \psi_1 - \psi_2$, $A = A_1 - A_2$. Then we have $\psi(0) = 0$ and $A(0) = 0$.

Proof of Theorem 2.1. It follows from (2.1) that

$$
\begin{aligned}
\eta \frac{\partial \psi}{\partial t} + \left(\frac{i}{\kappa} \mathrm{grad} + A_1 \right)^2 \psi_1 - \left(\frac{i}{\kappa} \mathrm{grad} + A_2 \right)^2 \psi_2 \\
= i\eta\kappa\psi_1 \,\mathrm{div}\, A_1 - i\eta\kappa\psi_2 \,\mathrm{div}\, A_2 + \left(1 - |\psi_1|^2\right)\psi_1 - \left(1 - |\psi_2|^2\right)\psi_2.
\end{aligned}
\tag{4.1}
$$

Due to

$$\left(\frac{i}{\kappa}\operatorname{grad} + A_1\right)^2 \psi_1 = -\frac{1}{\kappa^2}\Delta\psi_1 + \frac{2i}{\kappa}A_1\cdot\operatorname{grad}\psi_1 + \frac{i}{\kappa}\psi_1 \operatorname{div} A_1 + |A_1|^2\psi_1,$$

we see that

$$\begin{aligned}\eta\frac{\partial\psi}{\partial t} - \frac{1}{\kappa^2}\Delta\psi = {} & -\frac{2i}{\kappa}(A\operatorname{grad}\psi_1 + A_2\operatorname{grad}\psi) \\ & + \left(i\eta\kappa - \frac{i}{\kappa}\right)(\psi\operatorname{div}A_1 + \psi_2\operatorname{div}A) \\ & - (|A_1| + |A_2|)(|A_1| - |A_2|)\psi_1 \\ & - |A_2|^2\psi + \psi + (|\psi_1| + |\psi_2|)(|\psi_2| - |\psi_1|)\psi_2 - |\psi_1|^2\psi.\end{aligned} \tag{4.2}$$

Taking the real part of the inner product of (4.2) with ψ in $\mathscr{L}^2$, we find that

$$\begin{aligned}&\frac{1}{2}\eta\frac{d}{dt}\|\psi\|^2 + \frac{1}{\kappa^2}\|\operatorname{grad}\psi\|^2 \\ &\quad\le -\operatorname{Re}\int_\Omega \frac{2i}{\kappa}(A\operatorname{grad}\psi_1 + A_2\operatorname{grad}\psi)\psi^*\,dx \\ &\qquad + \operatorname{Re}\int_\Omega\left(i\eta\kappa - \frac{i}{\kappa}\right)(\psi\operatorname{div}A_1 + \psi_2\operatorname{div}A)\psi^*\,dx \\ &\qquad - \operatorname{Re}\int_\Omega(|A_1| + |A_2|)(|A_1| - |A_2|)\psi_1\psi^*\,dx \\ &\qquad + \|\psi\|^2 + \operatorname{Re}\int_\Omega(|\psi_1| + |\psi_2|)(|\psi_2| - |\psi_1|)\psi_2\psi^*dx.\end{aligned} \tag{4.3}$$

Since

$$\begin{aligned}\int_\Omega (A\operatorname{grad}\psi_1)\psi^*\,dx &= -\int_\Omega \psi_1\operatorname{grad}(A\psi^*)\,dx \\ &= -\int_\Omega \psi_1\psi^*\operatorname{div}A\,dx - \int_\Omega \psi_1 A\operatorname{grad}\psi^*\,dx,\end{aligned}$$

we see that

$$
\begin{aligned}
&\left| -\operatorname{Re} \int_\Omega \frac{2i}{\kappa} (A \operatorname{grad} \psi_1 + A_2 \operatorname{grad} \psi) \psi^* \, dx \right| \\
&\quad \le \frac{2}{\kappa} \bigg(\int_\Omega |\psi_1 \psi^* \operatorname{div} A| \, dx + \int_\Omega |\psi_1 A \operatorname{grad} \psi^*| \, dx \\
&\qquad\qquad + \int_\Omega |A_2 \psi^* \operatorname{grad} \psi| \, dx \bigg) \\
&\quad \le \frac{2}{\kappa} \|\psi_1\|_4 \|\psi\|_4 \|\operatorname{div} A\| + \frac{2}{\kappa} \|\psi_1\|_4 \|A\|_4 \|\operatorname{grad} \psi\| \\
&\qquad + \frac{2}{\kappa} \|A_2\|_4 \|\psi\|_4 \|\operatorname{grad} \psi\| \\
&\quad \le C \|\psi_1\|_4 \|\psi\|_{H^1}^{3/4} \|\psi\|^{1/4} \|A\|_{H^1} + C \|\psi_1\|_4 \|A\|_{H^1}^{3/4} \|A\|^{1/4} \|\operatorname{grad} \psi\| \\
&\qquad + C \|A_2\|_4 \|\psi\|_{H^1}^{3/4} \|\psi\|^{1/4} \|\operatorname{grad} \psi\| \\
&\qquad\qquad \text{(by the interpolation inequality)} \\
&\quad \le C \|\psi\|_{H^1}^{3/4} \|\psi\|^{1/4} \|A\|_{H^1} + C \|A\|_{H^1}^{3/4} \|A\|^{1/4} \|\operatorname{grad} \psi\| \\
&\qquad + C \|A_2\|_4 \|\psi\|_{H^1}^{7/4} \|\psi\|^{1/4} \qquad \text{(by Lemmas 3.1 and 3.3)} \\
&\quad \le \frac{\epsilon}{4} \|A\|_{H^1}^2 + C(\epsilon) \|\psi\|_{H^1}^{3/2} \|\psi\|^{1/2} + \frac{\epsilon}{4} \|\operatorname{grad} \psi\|^2 \\
&\qquad + C(\epsilon) \|A\|_{H^1}^{3/2} \|A\|^{1/2} + \frac{\epsilon}{4} \|\psi\|_{H^1}^2 + C(\epsilon) \|A_2\|_4^8 \|\psi\|^2 \\
&\qquad\qquad (\forall \epsilon > 0) \\
&\quad \le \epsilon \|A\|_{H^1}^2 + \epsilon \|\psi\|_{H^1}^2 + C(\epsilon) \big(1 + \|A_2\|_4^8 \big) \big(\|\psi\|^2 + \|A\|^2 \big) \\
&\quad \le \epsilon \|A\|_{H^1}^2 + \epsilon \|\psi\|_{H^1}^2 \\
&\qquad + C(\epsilon) \left(1 + \sum_{i=1}^{3} \|\operatorname{grad} |A_{2,i}|^{3/2} \|^2 \right) \big(\|\psi\|^2 + \|A\|^2 \big), \qquad (4.4)
\end{aligned}
$$

where we set $A_2 = (A_{2,1}, A_{2,2}, A_{2,3})$ and the last inequality follows from the analogue to (3.22). Since

$$
\begin{aligned}
&\operatorname{Re} \int_\Omega \left(i\eta\kappa - \frac{i}{\kappa} \right) (\psi \operatorname{div} A_1 + \psi_2 \operatorname{div} A) \psi^* \, dx \\
&\quad = \operatorname{Re} \int_\Omega \left(\eta\kappa - \frac{1}{\kappa} \right) i |\psi|^2 \operatorname{div} A_1 \, dx + \operatorname{Re} \int_\Omega \left(\eta\kappa - \frac{1}{\kappa} \right) i \psi_2 \psi^* \operatorname{div} A \, dx \\
&\quad = \operatorname{Re} \int_\Omega \left(\eta\kappa - \frac{1}{\kappa} \right) i \psi_2 \psi^* \operatorname{div} A \, dx,
\end{aligned}
$$

we get that

$$\left|\operatorname{Re}\int_\Omega\left(i\eta\kappa-\frac{i}{\kappa}\right)(\psi\operatorname{div}A_1+\psi_2\operatorname{div}A)\psi^*\,dx\right|$$

$$=\left|\operatorname{Re}\int_\Omega\left(\eta\kappa-\frac{1}{\kappa}\right)i\psi_2\psi^*\operatorname{div}A\,dx\right|\le\left|\eta\kappa-\frac{1}{\kappa}\right|\int_\Omega|\psi_2\psi^*\operatorname{div}A|\,dx$$

$$\le C\|\psi_2\|_4\|\psi\|_4\|\operatorname{div}A\|\le C\|\psi_2\|_4\|\psi\|_{H^1}^{3/4}\|\psi\|^{1/4}\|\operatorname{div}A\|$$

$$\le C\|\psi\|_{H^1}^{3/4}\|\psi\|^{1/4}\|\operatorname{div}A\|\le\epsilon\|\psi\|_{H^1}^2+\epsilon\|A\|_{H^1}^2+C(\epsilon)\|\psi\|^2,\quad(4.5)$$

$$\left|-\operatorname{Re}\int_\Omega(|A_1|+|A_2|)(|A_1|-|A_2|)\psi_1\psi^*\,dx\right|$$

$$\le\int_\Omega(|A_1|+|A_2|)|A|\,|\psi_1|\,|\psi|\,dx$$

$$\le(\|A_1\|_3+\|A_2\|_3)\|A\|_6\|\psi_1\|_4\|\psi\|_4\,dx$$

$$\le C\|A\|_{H^1}\|\psi\|_{H^1}^{3/4}\|\psi\|^{1/4}\qquad\text{(by Lemmas 3.1 and 3.3)}$$

$$\le\epsilon\|A\|_{H^1}^2+\epsilon\|\psi\|_{H^1}^2+C(\epsilon)\|\psi\|^2.\quad(4.6)$$

Also

$$\left|\operatorname{Re}\int_\Omega(|\psi_1|+|\psi_2|)(|\psi_2|-|\psi_1|)\psi_2\psi^*\,dx\right|$$

$$\le(\|\psi_1\|_4+\|\psi_2\|_4)\|\psi\|_4^2\|\psi_2\|_4$$

$$\le C\|\psi\|_4^2\le C\|\psi\|_{H^1}^{3/2}\|\psi\|^{1/2}\le\epsilon\|\psi\|_{H^1}^2+C(\epsilon)\|\psi\|^2.\quad(4.7)$$

By (4.3)–(4.7) we get that

$$\frac{1}{2}\eta\frac{d}{dt}\|\psi\|^2+\frac{1}{\kappa^2}\|\operatorname{grad}\psi\|^2$$

$$\le 4\epsilon(\|\operatorname{grad}\psi\|^2+\|\operatorname{grad}A\|^2)$$

$$+C(\epsilon)\left(1+\sum_{i=1}^3\|\operatorname{grad}|A_{2,i}|^{3/2}\|^2\right)(\|\psi\|^2+\|A\|^2).\quad(4.8)$$

From (2.2) we obtain that

$$\frac{\partial A}{\partial t}-\Delta A=\frac{1}{2i\kappa}(\psi^*\operatorname{grad}\psi_1+\psi_2^*\operatorname{grad}\psi-\psi\operatorname{grad}\psi_1^*-\psi_2\operatorname{grad}\psi^*)$$

$$-(|\psi_1|+|\psi_2|)(|\psi_1|-|\psi_2|)A_1-|\psi_2|^2A.\quad(4.9)$$

Taking the inner product of (4.9) with A in L^2, we see that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|A\|^2 + \|\text{grad } A\|^2 \\
&\quad = \frac{1}{2i\kappa}\int_\Omega (\psi^* \text{ grad } \psi_1 + \psi_2^* \text{ grad } \psi - \psi \text{ grad } \psi_1^* - \psi_2 \text{ grad } \psi^*) A \, dx \\
&\qquad - \int_\Omega (|\psi_1| + |\psi_2|)(|\psi_1| - |\psi_2|) A_1 A \, dx - \int_\Omega |\psi_2|^2 |A|^2 \, dx.
\end{aligned}$$

Majorizing each term above by similar methods used in (4.4)–(4.7), we infer that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|A\|^2 + \|\text{grad } A\|^2 \\
&\qquad \le \epsilon\left(\|\text{grad } \psi\|^2 + \|\text{grad } A\|^2\right) + C(\epsilon)\left(\|\psi\|^2 + \|A\|^2\right). \quad (4.10)
\end{aligned}$$

By (4.8) and (4.10) we get that

$$\begin{aligned}
&\frac{d}{dt}\left(\|\psi\|^2 + \|A\|^2\right) + \frac{2}{\eta\kappa^2}\|\text{grad } \psi\|^2 + 2\|\text{grad } A\|^2 \\
&\qquad \le 2\epsilon\left(1 + \frac{4}{\eta}\right)\left(\|\text{grad } \psi\|^2 + \|\text{grad } A\|^2\right) \\
&\qquad\quad + C(\epsilon)\left(1 + \sum_{i=1}^{3} \|\text{grad}|A_{2,i}|^{3/2}\|^2\right)\left(\|\psi\|^2 + \|A\|^2\right).
\end{aligned}$$

Choosing ϵ small enough, we see that

$$\frac{d}{dt}\left(\|\psi(t)\|^2 + \|A(t)\|^2\right) \le C\left(1 + \sum_{i=1}^{3} \|\text{grad}|A_{2,i}|^{3/2}\|^2\right)\left(\|\psi\|^2 + \|A\|^2\right).$$

By the Gronwall lemma, we infer that

$$\begin{aligned}
&\|\psi(t)\|^2 + \|A(t)\|^2 \\
&\qquad \le \left(\|\psi_0\|^2 + \|A_0\|^2\right) \exp\left(C\int_0^t \left(1 + \sum_{i=1}^{3} \|\text{grad}|A_{2,i}|^{3/2}\|^2\right) dt\right) \\
&\qquad = 0, \qquad \forall 0 \le t \le T \qquad (\text{by Lemma 3.3}),
\end{aligned}$$

which completes the proof.

We remark that the most difficult term in the above proof is

$$\int_\Omega |A_2 \psi^* \text{ grad } \psi| \, dx.$$

To estimate this term, we use the fact

$$\sum_{i=1}^{3} \|\text{grad}|A_{2,i}|^{3/2}\|^2 \in L^1(0,T).$$

If A_2 is more regular, then the proof is much easier. For example, if $A_0 \in L^4$, then we can use Lemma 3.4 and show Theorem 2.1 as follows.

Proof of Theorem 2.1 *with* $A_0 \in L^4$. At this moment, we only need to treat the term (see the proof of (4.4))

$$\int_\Omega |A_2 \psi^* \text{ grad } \psi| \, dx.$$

Obviously, we have

$$\begin{aligned}
&\frac{2}{\kappa}\int_\Omega |A_2 \psi^* \text{ grad } \psi| \, dx \\
&\quad \le \frac{2}{\kappa}\|A_2\|_4 \|\psi\|_4 \|\text{grad } \psi\| \\
&\quad \le C\|A_2\|_4 \|\psi\|_{H^1}^{3/4} \|\psi\|^{1/4} \|\text{grad } \psi\| \le C\|\psi\|_{H^1}^{3/4} \|\psi\|^{1/4} \|\text{grad } \psi\| \\
&\hspace{20em} \text{(by Lemma 3.4)} \\
&\quad \le \epsilon \|\psi\|_{H^1}^2 + C(\epsilon)\|\psi\|^2.
\end{aligned}$$

And thus repeating the above procedure, we can easily deduce that

$$\frac{d}{dt}\left(\|\psi(t)\|^2 + \|A(t)\|^2\right) \le C\left(\|\psi(t)\|^2 + \|A(t)\|^2\right),$$

and then Theorem 2.1 follows from the Gronwall lemma.

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